PHYSICAL REVIEW E **72**, 010301(R) (2005)

Spatially heterogeneous dynamics in a model for granular compaction

Alexandre Lefèvre,[1] Ludovic Berthier,[2] and Robin Stinchcombe[1]
[1]*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, United Kingdom*
[2]*Laboratoire des Verres UMR 5587, Université Montpellier II and CNRS, 34095 Montpellier, France*
(Received 29 October 2004; published 19 July 2005)

We suggest the emergence of spatially correlated dynamics in slowly compacting dense granular media by analyzing analytically and numerically multipoint correlation functions in a simple particle model characterized by slow nonequilibrium dynamics. We show that the logarithmically slow dynamics at large times is accompanied by spatially extended dynamic structures that resemble the ones observed in glass-forming liquids and dense colloidal suspensions. This suggests that dynamic heterogeneity is another key common feature present in very different jamming materials.

DOI: 10.1103/PhysRevE.72.010301 PACS number(s): 81.05.Rm, 05.70.Ln, 64.70.Pf

When a pile of grains is gently shaken, its volume fraction increases so slowly that the process hardly becomes stationary on experimental time scales [1,2]. This is reminiscent of the slow relaxation observed in glass formers, as noted long ago [3]. From a more fundamental point of view, it is tempting to build upon analogies and suggest that granular media, glasses, and other jamming systems can be described by common theoretical approaches [4]. In recent years, several aspects of glasses and granular media have been studied with similar approaches. Static structures have been studied to understand the relevance of the network of force chains between grains or atoms to the dynamics of the jammed state [5]. Also, since both grains and glasses undergo nonequilibrium "glassy" dynamics, the idea that an effective thermodynamics can be used has received considerable attention [6,7].

In this paper we also transfer knowledge from one field to another and show that the glassy dynamics of granular media is characterized by the appearance of spatiotemporal structures similar to the ones described as dynamic heterogeneity in glassformers [8] and dense colloidal suspensions [9]. Dynamic heterogeneity is believed to play a crucial role in the glass formation, and forms the core of recent theoretical descriptions [10,11]. Physically, it stems from the existence of spatial correlations in the local dynamics that extend beyond the ones revealed by static pair correlations. To study dynamic heterogeneity, correlators that probe more than two points in space and time have to be considered [8,10–12]. These spatial fluctuations have never been studied in models or experiments on granular compaction [13], although caged particle dynamics was recently studied in a sheared system [14]. Here we prove the emergence of large dynamic length scales in a particle model studied in detail in the context of granular compaction [15–19]. We take advantage of its relative simplicity to compute analytically multipoint correlations studied in glass formers and confirm our results by numerical simulations.

We consider the variant of the parking lot model [15] introduced in Ref. [16]. It is a one-dimensional process in which hard blocks of unit size are first irreversibly deposited at random positions on a line of linear size L until no place is available. Time scales are counted from the end of this deposition process, which corresponds to $t_w=0$. In a second step, particles are allowed to diffuse with a coefficient of diffusion D. The last dynamical rule consists of deposition events. When particles have diffused in such a way that a void of unit size opens, the void is instantaneously filled by a new particle and the density, $\rho(t_w)=N(t_w)/L$, increases by $1/L$, $N(t_w)$ being the number of particles present at time t_w. These rules lead to slow dynamics, because the larger the density the longer it takes to open a void. A more general version of this model includes evaporation [15,17]. It displays the generic features observed in granular compaction: logarithmic increase of the density [15–17], aging [17,19], non-Gaussian density fluctuations [18], effective temperatures, and link with Edwards entropy [17]. The time evolution of the density also describes the experiments,

$$\rho(t_w) = \rho_\infty - \left[a_0 + a_1 \ln\left(\frac{t_w - t_0}{\tau} \right) \right]^{-1}, \qquad (1)$$

where ρ_∞, a_0, a_1, τ, and t_0 are fitting parameters. For the present model without evaporation Eq. (1) holds at large t_w with $\rho_\infty=a_1=\tau=1$, $a_0=t_0=0$. Here we focus on local dynamic quantities and their spatial correlations through both analytic calculations and Monte Carlo simulations. The results of the simulations have been obtained by averaging over 2×10^4 independent histories with $L=250$. Time is measured in units of the diffusion constant, D, which is set to unity.

In Fig. 1 we illustrate the results reported in this paper, namely the emergence of spatiotemporal correlations in dynamic trajectories of compacting granular systems. In this figure, black (white) denote particles that are more (less) mobile than the average on a time scale of the order of the structural relaxation time. The time extension of the trajectory is 10^5, the spatial extension $L=500$, and the final density $\rho(10^5)\approx0.91$. The emergence of large-scale correlations in the local dynamics is evident, since domains that extend spatially over a hundred particles can be observed, despite the absence of any such large static correlations between particles. Spatial clustering of fast and slow regions is the hallmark of dynamic heterogeneity [8–10,12]. A closer look at the left part of the figure shows that the dynamic length scale is visibly smaller at shorter times, suggesting

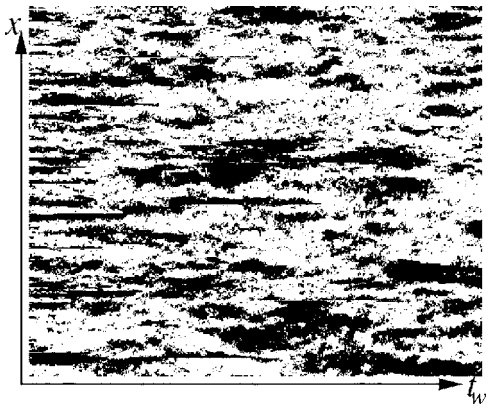

FIG. 1. Space-time pattern of dynamic fluctuations: particles are represented in black (white) when moving slower (faster) than average. Local dynamics is measured via the self-intermediate scattering function, Eq. (3), with $k=\pi$, at fixed $t-t_{\mathrm{w}}=2\times10^4$.

that spatial correlations are larger when the dynamics becomes slower, and possibly diverge in the jamming limit of a full system.

How did we build Fig. 1? As in supercooled liquids, information on local dynamics is accessed by following the dynamics of individual particles. Consider two times t_{w} and $t>t_{\mathrm{w}}\gg1$. The distribution of particle displacements is the self-part of the van Hove function,

$$G_s(r,t,t_{\mathrm{w}})=\frac{1}{N(t_{\mathrm{w}})}\sum_{i=1}^{N(t_{\mathrm{w}})}\langle\delta[r-\delta r_i(t,t_{\mathrm{w}})]\rangle, \tag{2}$$

where $\delta r_i(t,t_{\mathrm{w}})=r_i(t)-r_i(t_{\mathrm{w}})$ is the displacement of particle i between times t_{w} and t and only particles present at time t_{w} are summed over. Brackets indicate an average over initial conditions. Its Fourier transform is the self-intermediate scattering function

$$F_s(k,t,t_{\mathrm{w}})=\frac{1}{N(t_{\mathrm{w}})}\sum_{i=1}^{N(t_{\mathrm{w}})}\langle\cos[k\delta r_i(t,t_{\mathrm{w}})]\rangle. \tag{3}$$

Aging will manifest itself through an explicit dependence of $F_s(k,t,t_{\mathrm{w}})$ on its two-time arguments [6]. In the following we focus on the value $k=2\pi$, which corresponds to displacements of the order of the particle size. Since dynamic heterogeneity relates to spatial fluctuations about the averaged two-time dynamics, we have shown in Fig. 1 spatial fluctuations about $F_s(k,t,t_{\mathrm{w}})$. We colored black (white) those particles for which $\delta F_i(k,t,t_{\mathrm{w}})=\cos[k\delta r_i(t,t_{\mathrm{w}})]-F_s(k,t,t_{\mathrm{w}})$ is negative (positive), i.e., those particles that move more (less) than average in a particular realization.

To further quantify spatial correlations we consider the structure factor of the dynamic heterogeneity,

$$S_k(q,t,t_{\mathrm{w}})=\frac{1}{Nf_k}\sum_{l,m}^{N}\langle\delta F_l\delta F_m e^{iq[r_l(t_{\mathrm{w}})-r_m(t_{\mathrm{w}})]}\rangle, \tag{4}$$

where $f_k(t,t_{\mathrm{w}})=N^{-1}\Sigma_i\langle\delta F_i(k,t,t_{\mathrm{w}})^2\rangle$ normalizes the structure factor and some obvious time and wave-vector dependencies have been removed, for clarity. By definition, the structure factor is built from two-time two-point quantities, which is the minimum requirement to probe the spatiotemporal patterns of Fig. 1. The small q limit in Eq. (4) defines a dynamic "four-point" susceptibility,

$$\chi_k(t,t_{\mathrm{w}})=S_k(q=0,t,t_{\mathrm{w}}), \tag{5}$$

which can be rewritten as the variance of the fluctuations of the spatially averaged two-time dynamics. Physically, dynamic fluctuations increase when the number of independently relaxing objects decreases [12]. Normalizations ensure that χ_k is finite in the thermodynamic limit, except at a dynamic critical point [11].

The quantities introduced above can be estimated analytically as follows. Between two deposition events, the total free volume is $L(1-\rho)$ and thus the slow dynamics is equivalent to a continuum version of a symmetric exclusion process at density $\tilde{\rho}=\rho/(1-\rho)$, i.e., involving caged dynamics. This process is then mapped onto a fluctuating interface model as follows [20]: We introduce an "interface" position $r(x,t)$ such that $r(x=i,t)=r_i(t)$. Fluctuations around the uniform configuration or flat interface are then introduced, $h(x,t)=r(x,t)-x/\rho$. The variation $dh(x,t)=h(x+dx,t)-h(x,t)$ of the interface fluctuation for $1\ll dx\ll L$ is a sum of dx independent Poissonian variables with variance $\tilde{\rho}^{-2}$ so that $P[dh(x,t)]\sim\exp[-(dh)^2/(2dx/\alpha)]$ where we defined $\alpha=\tilde{\rho}^2$. This leads to the equilibrium distribution

$$P_{\mathrm{eq}}[\{h(x)\}]=\exp\left(-\frac{\alpha}{2}\int_{-\infty}^{\infty}dx[\nabla h(x)]^2\right), \tag{6}$$

associated to an Edwards-Wilkinson dynamics [21] of the interface deviation,

$$\frac{\partial h(x,t)}{\partial t}=\alpha\nabla^2h(x,t)+\eta(x,t), \tag{7}$$

where η is a Gaussian white noise with zero mean and variance $\langle\eta(x,t)\eta(x',t')\rangle=2\delta(x-x')\delta(t-t')$. We see that deviations from the averaged position of a particle arise with an elastic penalty in the interface representation. This elastic behavior makes the following calculation similar to the evaluation of the elastic contribution to the dynamical susceptibility (5) in supercooled liquids performed in Ref. [22].

Equation (7) is first solved in the Fourier space,

$$\hat{h}(q,t)=e^{-\alpha q^2\tau}\hat{h}(q,t_{\mathrm{w}})+\int_{t_{\mathrm{w}}}^{t}dt'\,\eta(q,t')e^{-\alpha(t-t')}, \tag{8}$$

where we have defined the time difference $\tau=t-t_{\mathrm{w}}$. A crucial approximation is made here since α in Eq. (8) has in fact a logarithmic time dependence. This amounts to neglecting the effect of further deposition events on the particles already present at t_{w}. Since deposition is such a rare event at large times this approximation should capture the evolution of the local dynamics, as our numerical simulations shall confirm. Using (8), one easily gets

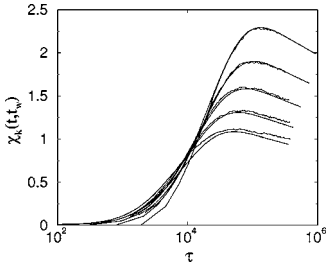

FIG. 2. Comparison of the numerical and analytical evaluations of $\chi_k(t,t_w)$ as a function of $\tau=t-t_w$ for $k=2\pi$ and $t_w=6250$, 12 500, 25 000, 50 000, 100 000 (from bottom to top).

$$F_s(k,t,t_w)=\exp\left(-k^2\sqrt{\frac{2\tau}{\alpha(t_w)}}g(0)\right), \tag{9}$$

where $g(x)=\int_{-\infty}^{\infty}dq e^{iqx}(1-e^{-q^2/2})/q^2$, so that $g(0)=(2\pi)^{1/2}$. Equation (9) is a classic result for the ordering process of one-dimensional random walkers [23]. It shows that $F_s(k,t,t_w)$ displays aging behavior and scales with $\tau/\tau_k(t_w)$, with a logarithmically increasing relaxation time scale, $\tau_k(t_w)\sim\alpha(t_w)/k^4$. At large times, $t-t_w\gg 1$, particles are subdiffusing, $\langle\delta r_i^2(t,t_w)\rangle\sim\sqrt{\tau/\alpha(t_w)}$, and the van Hove function (2) is a Gaussian. Gaussiannity is a consequence of the diffusive nature of the microscopic particle motion that causes the discrepancy with the non-Gaussian distributions of displacements recently reported experimentally for a bidimensional geometry [14].

The dynamic susceptibility (5) can also be computed,

$$\chi_k(t,t_w)=\int dx\frac{\cosh\left[\frac{2k^2}{\tilde{\rho}(t_w)}\sqrt{2\tau}g\left(\frac{x}{\tilde{\rho}(t_w)\sqrt{2\tau}}\right)\right]-1}{\cosh\left(\frac{2k^2}{\tilde{\rho}(t_w)}\sqrt{2\tau}g(0)\right)-1}$$

$$=\frac{\alpha(t_w)}{k^2}\mathcal{F}\left(\frac{\tau}{\tau_k(t_w)}\right), \tag{10}$$

where $\mathcal{F}(x)$ is a scaling function defined from the first line of Eq. (10). Careful analysis of Eq. (10) shows that $\chi_k(t,t_w)$ goes from zero at $\tau=0$ to the asymptotic value $\chi_k(t\to\infty,t_w)=\alpha(t_w)/(2k^2|g'(0)|)$, via a maximum $\chi_k^\star\sim\alpha(t_w)/k^2$ when $\tau\sim\tau_k(t_w)$. In Fig. 2 we show the dynamic susceptibility as a function of time separation τ for various ages t_w obtained in numerical simulations. The lines through the data are from Eq. (10). They are in excellent agreement with the data when t_w becomes large, justifying the approximation made above that the dynamics at large times is slow enough that deposition events have little influence on local dynamics.

The curves in Fig. 2 strikingly resemble the four-point susceptibilities discussed for supercooled liquids approaching their glass transition [8] and simple coarsening systems [12]. As for those systems, we conclude that dynamics is maximally heterogeneous when observed at time separations close to the relaxation time scale, itself dependent of the age of the sample. To our knowledge no experimental determination of the dynamic susceptibility $\chi_k(t,t_w)$ has been reported for granular media, although the experimental setup described in Ref. [14] would probably allow its determination.

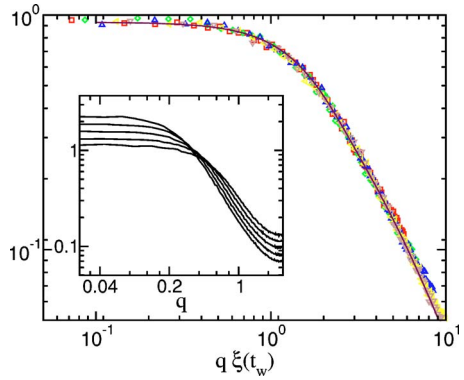

FIG. 3. (Color online) Inset: Simulated dynamic structure factor (4) as a function of q for parameters as in Fig. 2 and for $\tau=\tau_k(t_w)$. Main: Dynamic scale invariance, Eq. (11), is revealed by rescaling space by ξ_k and S_k by χ_k. The full line is the analytical result using the mapping to the Edwards-Wilkinson interface model.

The dynamics susceptibility $\chi_k(t,t_w)$ measures the volume integral of a spatial correlator. Therefore, an increasing susceptibility directly suggests the existence of a growing dynamic correlation length. This is most directly seen in the Fourier space when the wave-vector dependence of the dynamic structure factor (4) is considered. We have obtained an analytical form for $S_k(q,t,t_w)$ but it is too lengthy to be reported here. In Fig. 3 we compare this analytical expression for the wave-vector dependence of $S_k(q,t,t_w)$ for various t_w at time separations corresponding to the maximum of the dynamic susceptibility to numerical simulations. Again, the agreement between analytical and numerical results is excellent.

At fixed t_w the structure factor is characterized by a plateau at small q whose height is given by $\chi_k^\star(t_w)$. When q increases $S_k(q,t,t_w)$ leaves the plateau and decreases to 0 at large q. When t_w is increased the plateau becomes higher and it ends at a smaller wave vector but its overall shape is unchanged. This implies a dynamic scale invariance such as found in glass formers [10,11]: rescaling times by $\tau_k(t_w)$ and space by $\xi_k(t_w)\sim\alpha(t_w)/k^2$ makes trajectories statistically equivalent. Formally this means that the following scaling law is obeyed:

$$S_k(q,\tau_k,t_w)=\chi_k^\star(t_w)\mathcal{G}_k[q\xi_k(t_w)]. \tag{11}$$

Again we note that data in Fig. 3 strikingly resemble dynamic structure factors measured both in realistic supercooled liquids [8] and coarse-grained models for the glass transition [10].

A major result of the above analysis is the existence of a dynamic length scale, $\xi_k(t_w)$, which grows logarithmically with time when compaction proceeds, and therefore diverges when the systems jams. A diverging length scale provides support to the temporal renormalization group argument developed in Ref. [16], but we see no obvious connection be-

tween the dynamic criticality described here and the various power law scalings observed in static systems approaching jamming from above [24]. Physically, collective rearrangements of particles are needed to create a void of unit size, the denser the system the more cooperative the dynamics. A naive determination of ξ would rely on a free volume argument [25]. The mean free volume available to particles is $1/\tilde{\rho}$, so that the number of particles required to have a fluctuation of the free volume equal to unity is $\mathcal{N} \sim \tilde{\rho}^2 \sim \xi(t_w)$. This simple physical argument underlies the cooperative nature of the dynamics which is more formally captured by four-point correlation functions, Eqs. (4) and (5).

The idea that the size of collective motions increases when dynamics becomes slow is certainly not new. Multipoint correlation functions have now been measured in very different materials with similar qualitative results. That large dynamic length scales control glassy dynamics suggests the possibility that few universality classes underly and possibly unify the dynamical behaviour of a much wider diversity of jamming materials.

We thank S.N. Majumdar for enlightening discussions. A.L. is supported by Grant No. HPRN-CT-2002-00319 of STIPCO European Training Program.

[1] J. B. Knight, C. Fandrich, C. N. Lau, H. M. Jaeger, and S. R. Nagel, Phys. Rev. E **51**, 3957 (1995).
[2] P. Philippe and D. Bideau, Europhys. Lett. **60**, 677 (2002).
[3] L. C. E. Struik, *Physical Aging in Amorphous Polymers and Other Materials* (Elsevier, Amsterdam, 1978).
[4] A. J. Liu and S. R. Nagel, Nature (London) **396**, 21 (1998).
[5] C. S. O'Hern, S. A. Langer, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. **86**, 111 (2001).
[6] J.-P. Bouchaud, L. F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998).
[7] H. Makse and J. Kurchan, Nature (London) **415**, 614 (2002); G. D'Anna and G. Gremaud, *ibid.* **413**, 407 (2001).
[8] M. D. Ediger, Annu. Rev. Phys. Chem. **51**, 99 (2000); S. C. Glotzer, J. Non-Cryst. Solids **274**, 342 (2000).
[9] E. R. Weeks, J. C. Crocker, A. C. Levitt, A. Schofield, and D. A. Weitz, Science **287**, 627 (2000).
[10] J. P. Garrahan and D. Chandler, Phys. Rev. Lett. **89**, 035704 (2002); L. Berthier and J. P. Garrahan, Phys. Rev. E **68**, 041201 (2003).
[11] S. Whitelam, L. Berthier, and J. P. Garrahan, Phys. Rev. Lett. **92**, 185705 (2004).
[12] P. Mayer, H. Bissig, L. Berthier, L. Cipelletti, J. P. Garrahan, P. Sollich, and V. Trappe, Phys. Rev. Lett. **93**, 115701 (2004).
[13] J. J. Arenzon, Y. Levin, and M. Sellitto, Physica A **325**, 371 (2003).
[14] G. Marty and O. Dauchot, Phys. Rev. Lett. **94**, 015701 (2005).
[15] P. L. Krapivsky and E. BenNaim, J. Chem. Phys. **100**, 6778 (1994); X. Lin, G. Tarjus, and J. Talbot, J. Phys. A **27**, L195 (1994).
[16] R. Stinchcombe and M. Depken, Phys. Rev. Lett. **88**, 125701 (2002).
[17] M. Depken and R. Stinchcombe, e-print cond-mat/0404111.
[18] A. J. Kolan, E. R. Nowak, and A. V. Tkachenko, Phys. Rev. E **59**, 3094 (1999).
[19] J. Talbot, G. Tarjus, and P. Viot, Eur. Phys. J. E **5**, 445 (2001).
[20] S. N. Majumdar and M. Barma, Phys. Rev. B **44**, 5306 (1991).
[21] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).
[22] C. Toninelli, M. Wyart, L. Berthier, G. Biroli, and J.-P. Bouchaud, Phys. Rev. E **71**, 041505 (2005).
[23] T. E. Harris, J. Appl. Probab. **2** 323 (1965); R. Arratia, Ann. Prob. **11**, 362 (1983).
[24] C. S. O'Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. E **68**, 011306 (2003).
[25] T. Boutreux and P. G. de Gennes, Physica D **123**, 59 (1997).